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The Sharpness of Timan's Theorem on Differentiable Functions

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Our main result states that, given an increasing sequence a_n of positive numbers such that $\sum_{n=1}^{\infty} 1/na_n = \infty$, there exists a function f defined on $[0, 1]$ and not continuously differentiable on that interval such that $E_n(f) = O(1/n^2 a_n)$. This shows that a theorem of Timan cannot be improved.

I. INTRODUCTION

Let $C[a, b]$ denote the space of continuous real-valued functions defined on $[a, b]$, endowed with the uniform norm. If $f \in C[a, b]$, $E_n(f)$ denotes the distance from f to the subspace of algebraic polynomials of degree at most n. Let $C^1[a, b]$ be the subspace of $C[a, b]$ of continuously differentiable functions.

A classical theorem of Bernstein [1] states that f is continuously differentiable on the open interval (a, b) if $\sum_{n=1}^{\infty} E_n(f) < \infty$. Bernstein has also proved that this result is optimal in the sense that no matter how slowly the increasing sequence a_n tends to infinity, there exists $g \in C[a, b]$ with g' not continuous on (a, b) and such that $\sum_{r=1}^{\infty} E_n(g) = O(a_n)$.

Timan [7], [5, p. 74], and [8, p. 347], have proved the following:

THEOREM 1.1. *If ω is a modulus of continuity for which*

$$\sum_{n=1}^{\infty} \frac{1}{n} \omega \left(\frac{1}{n}\right) < \infty$$

and if, for $f \in C[-1, 1]$ and algebraic polynomials P_n of degree at most n, $n = 1, 2, 3,\ldots$,

$$|f(x) - P_n(x)| \leqslant \Delta_n(x)\, \omega(\Delta_n(x)|, \qquad -1 \leqslant x \leqslant 1, \tag{1}$$

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where

$$\Delta_n(x) = \max\left(\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right),$$

then $f \in C^1[-1, 1]$.

If $x \in (-1, 1)$, the hypothesis of Theorem 1.1 implies that $\sum_{n=1}^{\infty} |f(x) - P_n(x)| < \infty$, in accordance with Bernstein's theorem, whereas if x is one of the end points of the interval, (1.1) implies that

$$\sum_{n=1}^{\infty} n\,|f(x) - P_n(x)| < \infty. \tag{1.2}$$

It is the purpose of this article to show that this last condition cannot be weakened. More precisely, we shall prove

THEOREM 1.2. *Let a_n be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} 1/(na_n) = \infty$. Then there exists a function f in $C[0, 1]$ and not in $C^1[0, 1]$ such that*

$$E_n(f) = O\left(\frac{1}{n^2 a_n}\right). \tag{1.3}$$

An example of such a_n is $\prod_{i=1}^{k} \log^i n$, where $\log^i x$ denotes the ith iterate of $\log x$.

II. Proof of Theorem 1.2

In order to build the function f of Theorem 1.2, we need several preparatory results.

LEMMA 2.1. *Let $g(x)$ be a positive increasing continuous function defined for $x \geqslant 0$; let $a \geqslant e$. Then for $t \geqslant 0$, one has*

$$\int_t^{\infty} \frac{1}{a^x g(x)}\,dx \leqslant \frac{1}{a^t g(t)}.$$

The proof is immediate.

LEMMA 2.2. *Let $f \in C[a, b]$ and let $\alpha > 0$. Suppose that there exists a sequence of polynomials P_n such that*

$$\|P_n - f\| = O(1/n).$$

Then

$$\|P_n''\|_{[a+\alpha,b-\alpha]} = O(n).$$

The argument follows lines similar to those of the proof of Theorem 2.1 in [3], with Markoff's inequality replaced by Bernstein's inequality. For the sake of completeness we present the proof here: Let k be defined by $2^k \leqslant n < 2^{k-1}$. Then

$$P_n = P_n - P_{2^k} + \sum_{i=1}^{k} (P_{2^i} - P_{2^{i-1}}) + (P_1 - p_0) + P_0.$$

Bernstein's inequality gives

$$\|P_n''\|_{[a+\alpha,b-\alpha]} \leqslant K\left(n^2\|P_n - P_{2^k}\| + \sum_{i=1}^{k} 2^{2i}\|P_{2^i} - P_{2^{i-1}}\| + \|P_1 - P_0\|\right).$$

So that

$$\begin{aligned}
\|P_n''\|_{[a+\alpha,b-\alpha]} &\leqslant K\left(2n^2c/2^k + \sum_{i=1}^{k} 2^{2i-1}c/2^{i-1} + 2c\right)\\
&\leqslant K\left(2\cdot 2^{2(k+1)}c/2^k + \sum_{i=1}^{k} 2^{2i+1}c/2^{i-1} + 2c\right)\\
&\leqslant Kc\left(4\cdot 2^k + 4\sum_{i=1}^{k} 2^i + 2\right) \leqslant Mn.
\end{aligned}$$

The following result will play an important role in the proof of Theorem 1.2.

THEOREM 2.3. *Suppose that* $f \in C[-1,1]$ *satisfies the following properties*:

(a) *there exists a sequence of polynomials* P_n *such that*

$$\|P_n - f\| = o(1/n),$$

(b) $f'(0)$ *exists.*

Then $P_n'(0)$ *converges to* $f'(0)$.

Proof. By replacing $f(x)$ by $f(x) + ax + b$ we may assume that $f(0) = 0$ and $f'(0) = 1$. Suppose that, for some $\alpha > 0$, we have

$$P_n'(0) \geqslant 1 + \alpha, \qquad \text{i.o.} \tag{2.1}$$

(in this proof i.o. means for infinitely many values of n). By Lemma 2.2 and condition (a), $\|P_n''\|_{[-1/2,1/2]} \leqslant Kn$, so that

$$P_n'(0) - P_n'(x) \leqslant \frac{\alpha}{4}, \qquad x \in \left[0, \frac{\alpha}{4Kn}\right], \tag{2.2}$$

by the mean value theorem. It follows that, using (2.1) and (2.2),

$$P_n'(x) \geqslant 1 + \frac{3\alpha}{4}, \qquad x \in \left[0, \frac{\alpha}{4Kn}\right], \qquad \text{i.o.} \tag{2.3}$$

Now, for δ small enough,

$$f(x) \leqslant \left(1 + \frac{\alpha}{4}\right)x, \qquad x \in [0, \delta], \tag{2.4}$$

because $f'(0) = 1, f(0) = 0$. Suppose that $P_n(0) - f(0) = P_n(0) \leqslant \alpha^2/(16Kn)$. Then, using (2.3) and (2.4), we find

$$P_n\left(\frac{\alpha}{4Kn}\right) - f\left(\frac{\alpha}{4Kn}\right) \geqslant \frac{\alpha^2}{16Kn}, \qquad \text{i.o.},$$

and for n such that $\alpha/(4Kn) < \delta$. This contradicts (a). The argument is similar if we suppose that

$$P_n'(0) < 1 - \alpha, \qquad \text{i.o.}$$

Theorem 2.3 is proved.

The crucial step of Theorem 2.2 is the relation (2.2) which could not have been obtained by directly estimating P_n'. Indeed the proof of Lemma 2.2 shows that $\|P_n'\|_{[a+\alpha,b-\alpha]} \leqslant K \log n$ and this estimate is sharp. It can also be shown that, if the hypothesis $f'(0)$ exists and is deleted, the conclusion of Theorem 2.3 becomes $|P_n'(0)| = o(\log n)$.

Theorem 2.3 remains valid if the point 0 is replaced by any interior point of $[-1, 1]$. However, if a is one of the end points of the interval, $P_n'(a)$ converges to $f'(a)$ if $\|P_n - f\| = o(1/n^2)$. See also Theorems 2.4 and 2.5 in [3] for related results.

The idea of the above proof finds its origin in Theorem 2.5 in [4].

We now have built the necessary tools for the proof of the main result of this paper.

Proof of Theorem 1.2. Let

$$f(x) = \int_0^{\sqrt{x}} h(t)\, dt, \qquad 0 \leqslant x \leqslant 1,$$

where

$$h(t) = \sum_{r=1}^{\infty} \frac{1}{5^r a_{5^r}} T_{5^r}(t)$$

and $T_n(t) = \cos n \arccos t$ is the nth Chebycheff polynomial.

We first show that

$$E_n(f) = O\left(\frac{1}{n^2 a_n}\right). \tag{2.5}$$

Let $g(x) = f(x^2)$, $-1 \leqslant x \leqslant 1$, so that

$$g(x) = \int_0^{|x|} h(t)\, dt, \qquad -1 \leqslant x \leqslant 1.$$

If $x > 0$, $g'(x) = h(x)$ and $g'(x) = -h(-x)$ if $x < 0$. Because $T_n(t)$ is odd if n is odd [6, p. 5], $h(x)$ is odd, so that $g'(x) = h(x)$ for $0 < |x| < 1$. Because $T_n(0) = 0$ if n is odd, $h(0) = 0 = g'(0)$. Hence $g'(x) = h(x)$ for $-1 \leqslant x \leqslant 1$. Let

$$P_n(x) = \sum_{r=1}^{s} \frac{1}{5^r a_{5^r}} T_{5^r}(x) \tag{2.6}$$

for $n = 5^s, 5^{s+1}, \ldots, 5^{s+1} - 1$. We have

$$E_n(h) \leqslant \|P_n - h\|_{[-1,1]} = \sum_{r=s+1}^{\infty} \frac{1}{5^r a_{5^r}}. \tag{2.7}$$

(In fact, we have $E_n(h) = \|P_n - h\|$ by [8, p. 77].) Let $k(x)$, $x \geqslant 1$, be the piecewise linear continuous function whose knots are n, $n \geqslant 1$, and such that $k(n) = a_{5^n}$, $n \geqslant 1$. We obtain, using the relation between s and n, and Lemma 2.1,

$$E_n(h) \leqslant \int_s^{\infty} \frac{dx}{5^x k(x)} \leqslant \frac{1}{5^s a_{5^s}} \leqslant \frac{K}{n a_n}. \tag{2.8}$$

Using [5, p. 79], we obtain

$$E_n(g) \leqslant \frac{K_1}{n} E_{n-1}(g') = \frac{K_1}{n} E_{n-1}(h) \leqslant \frac{K_2}{n^2 a_n}. \tag{2.9}$$

Now let R_{2n} be the polynomial (of degree at most $2n$) of best approximation to g on $[-1, 1]$. But g is an even function. It follows that R_{2n} is even [5,

p. 34], so that $R_{2n}(x) = Q_n(x^2)$ for some polynomial Q_n. Because $g(x) = f(x^2)$, we obtain

$$\begin{aligned}\|R_{2n}(x) - g(x)\|_{[-1,1]} &= \|Q_n(x^2) - f(x^2)\|_{[-1,1]} \\ &= \|Q_n(x^2) - f(x^2)\|_{[0,1]} \\ &= \|Q_n(x) - f(x)\|_{[0,1]}. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$\|Q_n - f\|_{[0,1]} = O\left(\frac{1}{n^2 a_n}\right).$$

(1.3) is established. It remains, in order to complete the proof of Theorem 1.2, to show that $f \notin C^1[0, 1]$. We remark first that $f'(x)$ exists and is continuous for $1 < x \leqslant 1$. Suppose that $\lim_{x \to 0^+} f'(x)$ exists. Because $h(0) = 0$, we would have

$$2 \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{h(\sqrt{x})}{\sqrt{x}} = \lim_{x \to 0^+} \frac{h(x)}{x} = h'_+(0).$$

But h is odd as already noticed, so that $h'_-(0)$ would exist and be equal to $h'_+(0)$. We show now that this is impossible by proving that $h'(0)$ does not exist. Indeed, on the one hand, there exists, by (2.8), a sequence P_n of polynomials such that

$$\|P_n - h\|_{[-1,1]} = o\left(\frac{1}{n}\right). \tag{2.11}$$

(We suppose, without loss of generality, that $\lim_{n \to \infty} a_n = \infty$.) On the other hand, we obtain, using the definition of P_n given in (2.6) and the fact that $T'_{5^k}(0) = 5^k$ (because $5^k \equiv 1 \bmod 4$),

$$\begin{aligned} P'_{5^n}(0) &= \sum_{r=1}^{n} \frac{1}{5^r a_{5^r}} T'_{5^r}(0) \\ &= \sum_{r=1}^{n} \frac{1}{a_{5^r}}. \end{aligned}$$

So that

$$\lim_{n \to \infty} P'_{5^n}(0) = \infty \tag{2.12}$$

by the Cauchy condensation theorem and the hypothesis on a_n. (2.11), (2.12) and Theorem 2.2 show that $h'(0)$ does not exist. That shows that $f \notin C^1[0, 1]$.

The proof of Theorem 1.2 is complete.

It would be of interest to see whether or not the above function f belongs to the Lip 1 class of $[0, 1]$. Our attempts to answer this question have been unsuccessful.

It follows from a result of Zygmund [9, p. 48] that the function h considered in the proof of Theorem 1.2 is differentiable in a set of the power of the continuum in every interval. The proof that $h(x)$ is not differentiable at 0 relies on Theorem 2.3, which is based on Lemma 2.2, which, in turn, is proved by using Bernstein's inequality. It does not seem that the techniques used by Hardy in [2] to prove that $\sum_{n=0}^{\infty} 1/a^n \cos a^n x$, $a > 1$, is nowhere differentiable may be applied to show that $h'(0)$ does not exist.

We end this section by noting that Jackson's theorem [5, pp. 66, 67] yields only that $E_n(g) = o(1/n)$ if $g \in C^1[0, 1]$, whereas Theorem 1.2 shows the existence of a non-C^1 function f for which $E_n(f) = O(1/n^2 a_n)$.

III. Remarks and an Open Problem

In order for the function f defined on $[-1, 1]$ to be continuously differentiable, the theorem of Timan (Theorem 1.2) requires that $|P_n(x) - f(x)| \leqslant 1/n^2\, \omega(1/n^2)$ with $\sum_{n=1}^{\infty} 1/n\, \omega(1/n) < \infty$ if $x = \pm 1$.

The following theorem shows that a sufficient condition for f to be in $C^1[a, b]$ is $\sum_{n=1}^{\infty} nE_n(f) < \infty$. This result slightly improves Timan's theorem if x is one of the end points in the interval in the sense that $E_n(f)$ is not required to decrease like $1/n\, \omega(1/n^2)$. (Of course Theorem 3.2 is much weaker than Timan's theorem in the interior of the interval.)

Although we believe that the following extension of the Cauchy condensation test is known, we are not aware of any reference to it.

LEMMA 3.1. *Let a_n be a decreasing sequence of positive numbers. If*

$$\sum_{n=1}^{\infty} na_n < \infty$$

then

$$\sum_{n=1}^{\infty} 2^{2n} a_{2^n} < \infty.$$

Proof. The lemma follows from

$$\sum_{k=1}^{r} 4^k a_{2^k} \leqslant 4 \sum_{k=1}^{2^r} k a_k, \qquad r = 1, 2, \ldots, \tag{3.1}$$

which we will proceed to establish. (3.1) is true for $r = 1$ and suppose that it holds for $r = n$. Then the proof of (3.1) reduces to showing that

$$4^n a_{2^{n+1}} \leqslant \sum_{k=2^n+1}^{2^{n+1}} k a_k.$$

Now the hypothesis on a_k yields

$$\begin{aligned}
\sum_{k=2^n+1}^{2^{n+1}} k a_k &\geqslant a_{2^{n+1}} \sum_{k=2^n+1}^{2^{n+1}} k \\
&= a_{2^{n+1}} \frac{1}{2} ((2^{n+1} + 1)\, 2^{n+1} - (2^n + 2)(2^n + 1)) \\
&= 4^n a_{2^{n+1}} \left(2 + \frac{1}{2^n} - \frac{1}{2} - \frac{3}{2^{n+1}} - \frac{1}{2^{2n}}\right) \\
&\geqslant 4^n a_{2^{n+1}}, \qquad n \geqslant 1.
\end{aligned}$$

The lemma is proved.

If na_n were decreasing, the conclusion of Lemma 3.1 would follow immediately from the Cauchy condensation test. In the following theorem, the above lemma will be used with $a_n = E_n(f)$. Because $E_n(f)$ may remain constant for an arbitrarily large number of (consecutive) values of n [8, p. 40], $nE_n(f)$ need not be decreasing.

THEOREM 3.2. *Let $f \in C[a,b]$. If*

$$\sum_{n=1}^{\infty} nE_n(f) < \infty$$

then

$$f \in C^1[a, b].$$

The proof is an application of the classical telescopic technique of Bernstein, Markoff's inequality and the above lemma: Let P_n be the polynomial of best approximation to f on $[a, b]$ and let

$$S_0 = P_1, \qquad S_n = P_{2^n} - P_{2^{n-1}},$$

so that

$$f(x) = \sum_{n=0}^{\infty} S_n(x)$$

uniformly on $[a, b]$. Now Markoff's inequality gives

$$\begin{aligned} \|S_n'\| &\leqslant M2^{2n} \|S_n\| \\ &\leqslant M2^{2n}(\|P_{2^n} - f\| + \|P_{2^{n-1}} - f\|) \\ &\leqslant M2^{2n+1} E_{2^{n-1}}(f). \end{aligned}$$

But the hypothesis on $E_n(f)$ and Lemma 3.1 yield

$$\sum_{n=0}^{\infty} 2^{2n+1} E_{2^{n-1}}(f) < \infty.$$

It follows that $\sum_{n=0}^{\infty} S_n'(x)$ converges uniformly on $[a, b]$ (necessarily to $f'(x)$).

We can now combine Theorems 1.2 and 3.2 in

THEOREM 3.3. *An optimal sufficient condition for $f \in C[a, b]$ to be continuously differentiable is $\sum_{n=1}^{\infty} nE_n(f) < \infty$.*

The next result is proved by following lines similar to the proof of Theorem 3.2.

THEOREM 3.4. *Let $f \in C[a, b]$. If*

$$\sum_{n=1}^{\infty} n^{2k-1} E_n(f) < \infty$$

then

$$f \in C^k[a, b],$$

where k is a positive integer.

We believe that, as in Theorem 3.2, Theorem 3.4 is optimal in the following sense.

Conjecture 3.5. Let a_n be an increasing sequence of positive integers such that $\sum_{n=1}^{\infty} 1/na_n \leqslant \infty$. Then there exists a function f in $C[0, 1]$ but not in $C^k[0, 1]$ such that

$$E_n(f) = O\left(\frac{1}{n^{2k} a_n}\right) \qquad (k \geqslant 2).$$

Our attempts to build such a function have been unsuccessful. We were only able to prove the following result, which first require some notations: We say that a function g defined on $[a, b]$ satisfies a Lipschitz condition of order α if $|g(x)-g(y)| \leqslant M|x-y|^\alpha$, $x, y \in [a, b]$; and we write $g \in \operatorname{Lip} \alpha$.

THEOREM 3.6. *For every positive integer k and for every $0<\alpha<1$, there exists a function $f \in C[0, 1]$ such that, for $n=1, 2, 3, \ldots$,*

$$E_n(f) \leqslant \frac{M_k}{n^{2(k+\alpha)}}$$

and such that

$$\frac{R_k}{n^\alpha} \leqslant \omega\left(f^{(k)}, \frac{1}{n}\right) \leqslant \frac{S_k}{n^\alpha}.$$

COROLLARY 3.7. *For every positive integer r and for every $0<\beta<1$, there exists a function $f \in C[0, 1]$, $f \notin C^r[0, 1]$ and*

$$E_n(f) = O\left(\frac{1}{n^{2r-\beta}}\right).$$

Proof of the Corollary. Apply Theorem 3.6 with $r=k-1$ and $\beta = 1-\alpha/2$.

The gap between Corollary 3.7 and Conjecture 3.5 is clear.

Proof of Theorem 3.6. Let $f(x)=x^{k+\alpha}$, $0 \leqslant x \leqslant 1$. Consider the function $g(x)=x^{2k}|x|^{2\alpha}$, $-1 \leqslant x \leqslant 1$. It is easy to see that $g^{(2k)}$ exists and belongs to the class Lip 2α if $\alpha \leqslant \frac{1}{2}$ and that $g^{(2k+1)}$ exists and belongs to Lip $2\alpha-1$ if $\alpha > \frac{1}{2}$ (and <1). It follows from Jackson's theorem that $E_n(g) \leqslant K/n^{2k+2\alpha}$, $0<\alpha<1$. (In fact $1/n^{2k+2\alpha}$ is the exact order of decrease of $E_n(g)$. See [8, p. 412].) Let, as in the proof of Theorem 1.2, P_{2n} be the polynomial of best approximation of degree at most $2n$ to g on $[-1, 1]$. Because P_{2n} is even, g being even, it is of the form $P_{2n}(x)=Q_n(x^2)$. Now, for a fixed k,

$$\begin{aligned}\frac{K}{n^{2k+2\alpha}} &\geqslant \|Q_n(x^2)-x^{2k}|x|^{2\alpha}\|_{[-1,1]}\\ &= \|Q_n(x^2)-x^{2k}x^{2\alpha}\|_{[0,1]}\\ &= \|Q_n(x)-x^k x^\alpha\|_{[0,1]}.\end{aligned}$$

And, on the other hand,

$$\frac{K_1}{n^\alpha} \leqslant \omega\left(f^{(k)}, \frac{1}{n}\right) \leqslant \frac{K_2}{n^\alpha},$$

where $f(x) = x^{k+\alpha}$. The theorem is proved.

It is worth noticing that the theorem of Jackson (or of Timan [5, p. 66]), applied directly to f, yields only that $E_n(f) \leqslant K_3/n^{k+\alpha}$.

References

1. S. N. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné, *Mém. Acad. Roy. Belgique* **4** (2), (1912), 1–103.
2. G. H. Hardy, Weierstrass' non-differentiable function, *Trans. Amer. Math. Soc.* **17** (1916), 301–325.
3. M. Hasson, Derivatives of the algebraic polynomials of best approximation, *J. Approx. Theory* **29** (1980), 91–102.
4. M. Hasson, Comparison between the degrees of approximation by lacunary and ordinary algebraic polynomials, *J. Approx. Theory* **29** (1980), 103–115.
5. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
6. T. J. Rivlin, "The Chebyshev Polynomials," Wiley, New York, 1964.
7. A. F. Timan, Converse theorems of the constructive theory of functions defined on a finite segment of the real axis, *Dokl. Akad. Nauk SSSR* **116**. No. 5 (1957), 762–765.
8. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Macmillan Co., New York, 1963.
9. A. Zygmund, "Trigonometric Series," Vol. 1, 2nd ed., Cambridge Univ. Press, London, 1959.